MEMOIRS, &c.

PART I.

MATHEMATICAL AND ASTRONOMICAL PAPERS.

I. *Geometrical Methods of finding any required Series of Mean Proportionals between given Extremes, by* JAMES WINTHROP, *Esq.* F. A. A.

THE duplication of the cube has long been a desideratum in Geometry, and has been considered as one of the greatest questions in that science. To enhance its importance, its origin has been traced to the Oracle of Delphos. Hence it is frequently called the Delphick question. It is said, that when a pestilence raged in Athens, application was made to Apollo for information of the means, by which the anger of the Deity might be appeased. The God directed the inquirers to double the contents of his cubical altar, without changing its form. He was, however, mistaken in his prescription; for the plague ceased, though the condition was not complied with. The writer of the present paper therefore, requests the candour of this learned assembly, whilst he attempts to apply to this problem, principles universally received.

It is a well known property of proportionality, that if a ſeries of four continued proportionals be taken, the firſt term will be to the fourth, as the cube of the firſt is to the cube of the ſecond. The queſtion therefore has been made to change ground; and the object has been, to find two geometrical proportionals between two given extremes. This is the ſituation in which it is to be viewed at preſent.

That triangles, agreeing in all their angles, have their correſponding ſides proportional, is one of thoſe truths which enter into almoſt every geometrical demonſtration; and it is the baſis of this inveſtigation.

All caſes of this ſort may be claſſed under two general heads. I. When the whole number of terms is uneven. II. When the whole number of terms is even.

For the greater perſpicuity, let us begin with the moſt ſimple caſes, which, though the methods of reſolving them are known, muſt form the baſis of our reaſoning; and, without which, the ſyſtem will not be complete. The whole will be compriſed in four propoſitions.

PROBLEM I.

IT is required to find one mean proportional between two given extremes.

It is a well known property of a circle, that a ſine is always a geometrical mean between the verſed ſine and the remaining part of the diameter. Therefore, make a right line A B (ſee Plate 1. Fig. 1.) equal to the ſum of the given extremes A C and C B; from the point of union C, raiſe a perpendicular

perpendicular C D, and bifecting the line A B in E, with the radius A E draw the femicircle A F B; then will the line C F, intercepted between the circumference and the diameter of the femicircle, be the mean term fought.

PROBLEM II.

TO find two mean proportionals between two given extremes.

This muft be done by fimilar triangles, it being an univerfal principle, that fimilar triangles have proportional fides. Let A C E, E C D, and D C B, (Fig. 2.) be equal angles of any affumed magnitude; and let A C and B C be the extremes given. Draw A B, a right line croffing E C and D C in the points E and D. Then will A E C equal E D C+D C E, and for the fame reafon C B D will equal C D E—D C E. Wherefore, make B D F equal to D C E, and A E G equal to D C E, and we fhall have three fimilar triangles F C D, D C E, and E C G, and their fides are neceffarily proportional. And the lines C F, C D, C E, and C G form a feries of four continued proportionals: For C D is the hypothenufe of the firft triangle, and bafe of the fecond; and is therefore a mean between C F and C E. In like manner C E is the hypothenufe of the fecond, and the bafe of the third triangle; and is therefore, a mean between C D and C G. But the extremes C F and C G are fhorter than C B and C A. Having therefore, by this procefs, afcertained the method of finding eafily four continued proportionals, by beginning with the mean terms; if we invert the procefs, and begin with the extremes, and make the angles E A I and

D B K each equal to D C E, we ſhall have B K parallel to F D, and A I parallel to E G, and therefore K I parallel to D E. Therefore, the triangles C B K, C K I, and C I A are ſimilar to C D E; and by reaſon of poſition, C K and C I are the mean proportionals ſought. It is evident, that the two triangles C F D and C E G are introduced only for the demonſtration; but in practice, it is only neceſſary, after joining the points A and B, to make E A I equal to D C E and we have the term in the ſeries next to one of the extremes, which is all that is in ſuch a caſe required. Clearneſs made it neceſſary in the preſent inſtance, that the whole proceſs ſhould be performed.

PROBLEM III.

TO find any required even number of mean continued proportionals between two given extremes.

From what was ſaid, under the ſecond propoſition, it is evident, that if any number of ſimilar triangles be made, and the ſubtenſe of one of the angles at C (Fig. 3.) be continued as E E, the angle E E I will always equal E C E. The angle B A E is therefore, always equal to all the angles on one ſide of the middle angle, when the number of angles at C is uneven, as it is in the preſent problem. The number of angles at C is always leſs by one, than the number of terms. If therefore, the number of terms be twelve, the angles will be eleven; and B A E will equal five of them, being all the angles on one ſide of the middle angle. C E, the ſecond line in the figure, is therefore, one of the lines ſought, and next in order to one of the extremes, from which all the reſt are eaſily obtained.

PROBLEM

PROBLEM IV.

TO find any uneven number of mean continued proportionals, the extremes being given.

Here, the angles at C (Fig. 4.) being even, after preparing the work according to Fig. 2, it becomes neceſſary to find the general mean of the whole ſeries, by Prob. I. and to mark it off from C to D upon the middle line. Draw A D and B D. Then if B D be continued to E, the angle A D E will, by conſtruction, be equal to A C D, equal to all the angles on one ſide of the general mean C D. Therefore make E D F equal to one of the angles at C, and mark the point F in the next line, in the ſeries following C D; and the diſtance C F will be the length of the term in the ſeries next following the general mean.

This ſyſtem is now reduced to a ſimple form, and applies to all caſes. Inſtead of its remaining any longer a difficulty to double ſolids, it appears, that they may be multiplied in any ratio whatever, by an eaſy proceſs. It is to be obſerved, that when the number of terms is even, the term firſt found is next to one of the extremes; and when the whole number is uneven, it is next to the general mean. I ſhall be gratified to find hereafter, that the method is ſimplified ſtill further, by a happier hand than mine.